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Tunneling Effect and Hawking Radiation from a Gibbon–Maeda Black Hole by Using Eddington–Finkelstein Coordinates

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Abstract In this paper, by using well-known Eddington–Finkelstein coordinates instead of Painlevè coordinates, we study the tunneling effect of black holes. As examples of special static black holes, we calculate the tunneling rates of Gibbon–Maeda black holes. The result obtained by adopting Eddington–Finkelstein coordinates is in agreement with the Parikh's standard result, $\Gamma \sim \exp(-2\,\mathrm{Im}\,S)$, which adopts the Painlevè coordinates. In addition, we discuss carefully the condition that the coordinates system in which we study the tunneling process should satisfy. In our opinion, the terms of the tunneling effect are not as strict as ones in Parikh's paper and could be softened properly.

1 Introduction

In 1970s, Stephen Hawking discovered that basic principles of quantum field theory lead to the emission of thermal radiation from a classical black hole [1], which gives rise to a famous paradox—the information loss paradox in black hole physics. Recently, Parikh and Wilczek gave an enlightening suggestion that Hawking radiation could be treated as a tunneling precess [2–4]. They obtained a leading correction to the emission rate arising from loss of mass of the black hole. Following this method, Hemming and Keski-Vakkuri have investigated the radiation from Ads black holes [5], and Medved has studied those from a de Sitter cosmology [6]. Zhang and Zhao have extended Parikh's method from static black holes to the non-spherical symmetric stationary black holes and radiation of charged particle and massive particle and made much progress [7–10]. However, in all these investigations, the coordinates used to investigate tunneling effect are Painlevè coordinates.

There are two difficulties to overcome in the calculation of the tunneling rate. The first is that there do not seem to be any barrier. The second is that in order to do a tunneling computation, one requires to find a coordinate system which is well-behaved at the event horizon.

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To overcome the first difficulty, they think that the barrier is created by the outgoing particle itself and energy (ADM energy) must be conserved. In order to overcome the second difficulty, Parikh introduces the Painlevè coordinates [3], $t = t_s + 2\sqrt{2mr} + 2m \ln \frac{\sqrt{r}-\sqrt{2m}}{\sqrt{r}+\sqrt{2m}}$, where t_s is Schwarzschild time and t is Painlevè time. The Painlevè metric has many attractive features. Firstly, the components of the metric in Painlevè coordinates are regular, not diverge at the event horizon. Secondly, constant-time slices are just flat Euclidean space. In paper [2], this feature is rather important because the WKB approximation is applied to calculate the tunneling rate. WKB approximation is derived from the quantum mechanics which is right in flat space. Thirdly, there exists a time-like killing vector field, which is important to the energy conservation. Finally, because in quantum mechanics, particle tunneling a barrier is a instantaneous process, Zhao and Zhang [11–13] suggest that the metric in the coordinates should satisfy Landau' coordinate clock synchronization condition [14]. Fortunately, the metric in the Painlevè coordinates does satisfy Landau' coordinate clock synchronization condition.

In our opinion, the main aim and the crucial point to introduce a new coordinates system is to eliminate the singularity of the components of the metric at the event horizon. It is not only the Painlevè coordinates which satisfy the condition above. The well-known Eddington–Finkelstein coordinate, $v = t + r_*$, is rather suitable to study the tunneling effect too, where r_* is the tortoise coordinate. When we use the Eddington–Finkelstein coordinates instead of the Schwarzschild coordinates, the components of the metric are not singular too. So, we believe that we could study the tunneling process adopting Eddington–Finkelstein coordinates. Using this coordinate, we once calculated the tunneling rate of the Schwarzschild and the Kerr black holes [15, 16]. The results are rather successful and in agreement with Parikh' [2] and Zhang' [11] results respectively. Eddington–Finkelstein coordinates have some merits too. The line elements of the black holes in Eddington–Finkelstein coordinates are more simple than the line elements in Painlevè coordinates, which make us calculate the tunneling rate more easily. It is interesting that the constant-time slices in Eddington–Finkelstein coordinates are not flat Euclidean space to the GM black holes. According to the condition that the constant-time slices should be flat Euclidean space [3], it seems that the WKB approximation could not be used to study the tunneling process in Eddington–Finkelstein coordinates. We, however, will calculate the tunneling rate of the Gibbons–Maeda black hole and obtain the correct results. The reasonable explanation is that the WKB approximation can be extended to the space which is not flat Euclidean, although the WKB approximation is derived from the quantum mechanics. So, in our opinion, the conditions which the coordinate system should satisfy in Parikh' paper is too strict and could be softened properly. In addition, another important condition that the coordinates system should satisfy in Parikh–Wilczek's framework is that the event horizon and the time-like limit surface should coincide because we use WKB formula when we calculate the tunneling rate. The WKB approximation can be used only when the language of a point particle is appropriate at the event horizon. Because the infinite blue-shift takes place near the time-like limit surface, the characteristic wavelength of any wave packet is always arbitrarily small there and the geometrical optics limit becomes an especially reliable approximation. So, the event horizon and the time-like limit surface should be identical as we study the tunneling effect. In addition, the condition that the event horizon and the time-like limit surface should coincide warrants that the first order pole is on the event horizon, not on the time-like limit surface in the calculation of the imaginary part of the action.

The paper is organized as follows. In Sect. 2, we calculate the tunneling rate of the Gibbons–Maeda black holes. In Sect. 3, we will discuss the conditions which the investigated tunneling effect should satisfy. Throughout the paper, the units $G = c = \hbar = k_B = 1$ are used.

2 Tunneling Effect from Gibbons–Maeda Black Holes

The line element of Gibbons–Maeda black hole is described by

$$ds^2 = -\frac{(r-r_+)(r-r_-)}{R^2}dt^2 + \frac{R^2}{(r-r_+)(r-r_-)}dr^2 + R^2 d\Omega^2, \tag{1}$$

where $R^2 = r^2 - D^2$, r_+ and r_- are the radiuses of the event horizon and inner horizon, respectively, satisfy

$$r_\pm = m \pm \sqrt{m^2 + D^2 - P^2 - Q^2}, \tag{2}$$

and the axion dilaton charge D is

$$D = \frac{P^2 - Q^2}{2M}, \tag{3}$$

where P and Q are the magnetic and electric charge, respectively. m is ADM mass of the space-time. To describe tunneling process, it is necessary to choose coordinates which are not singular at the event horizon. We choose Eddington–Finkelstein coordinate, $v = t + r_*$, not the Painlevè coordinates, to study the tunneling effect, where r_* is tortoise coordinate. In order to obtain the line elements in the Eddington coordinate, we rewritten (1) as

$$ds^2 = \frac{(r-r_+)(r-r_-)}{R^2}\left[-dt^2 + \frac{R^4}{(r-r_+)^2(r-r_-)^2}dr^2\right] + R^2 d\Omega^2. \tag{4}$$

Let

$$dr_* = \frac{R^2}{(r-r_+)(r-r_-)}dr. \tag{5}$$

With this choice, the line element reads

$$ds^2 = -\frac{(r-r_+)(r-r_-)}{R^2}dv^2 + 2dvdr + R^2 d\Omega^2. \tag{6}$$

Comparing with the line elements in Painlevè coordinates

$$ds^2 = -(1-b)\Delta dT^2 + 2\sqrt{b}dTdr + \frac{1}{\Delta}dr^2 + R^2 d\Omega^2, \tag{7}$$

where T is Painlevè time, defined by $dT = dt + \frac{1}{\Delta}\frac{\sqrt{b}}{1-b}dr$, the line elements (6) are more simple, which make the calculation more easily, where $b = \frac{r_+}{r_-}$, $\Delta = \frac{r^2(1-\frac{r_-}{r})}{R^2}$. It is obvious that the components are not diverge at the event horizon in (6) and $(\frac{\partial}{\partial v})^a$ is a time-like vector field. Note that the constant-time slices of the line elements in the Eddington–Finkelstein coordinates are not flat Euclidean space. The calculation below will show that this is not important to study the tunneling process.

The radial null geodesics in Eddington–Finkelstein coordinates obey

$$\dot{r} \equiv \frac{dr}{dv} = \frac{1}{2}\frac{(r-r_+)(r-r_-)}{R^2}. \tag{8}$$

Equations (6) and (8) are modified when the self-gravitation of the particle is considered. We could consider the particle with energy ω as a shell of energy. We fix the total mass (ADM mass) and allow the hole mass to fluctuate. When the shell of energy ω travels on the geodesics, we should replace m with $m-\omega$ in the geodesic equation (8) and in the line elements equation (6) to describe the moving of the shell [4].

In our picture, a point particle description is appropriate. Because of the infinite blue shift near the horizon, the characteristic wavelength of any wave packet is always arbitrarily small there, so that the geometrical optics limit becomes an especially reliable approximation. The geometrical limit allows us to obtain rigorous results directly in the language of particles, rather than having to use the second-quantized Bogolubov method. In fact, the point particle description here demands that the event horizon and the time-like limit surface coincide because the infinite blue shift occur only near the time-like limit surface. Fortunately, the two surfaces are identical to the GM black hole in Eddington–Finkelstein coordinates. In the semiclassical limit, we could apply the WKB formula. This relates the tunneling amplitude to the imaginary part of the particle action at stationary phase. The emission rate, Γ, is the square of the tunneling amplitude [3]:

$$\Gamma \sim \exp(-2\,\mathrm{Im}\,S). \tag{9}$$

The imaginary part of the action for an outgoing positive energy particle which crosses the horizon outwards from r_{in} to r_{out} could be expressed as

$$\mathrm{Im}\,S = \mathrm{Im}\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}} p_r dr = \mathrm{Im}\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}}\int_0^{p_r} dp'_r dr \tag{10}$$

where p_r is canonical momentum conjugate to r, $r_{\mathrm{in}} = r_+$ is the initial radius of the black hole, and $r_{\mathrm{out}} = r'_+$ is the final radius of the hole, where $r'_+ = r_+(m-\omega)$. We substitute Hamilton' equation $\dot{r} = \frac{dH}{dp_r}|_r$ into (10), change variable from momentum to energy, and switch the order of integration to obtain

$$\mathrm{Im}\,S = \mathrm{Im}\int_m^{m-\omega}\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}}\frac{dr}{\dot{r}}dH = \mathrm{Im}\int_0^{\omega}\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}}\frac{2R'^2 dr}{(r-r'_+)(r-r'_-)}(-d\omega'), \tag{11}$$

where $R' = R(m-\omega)$. We have used the modified equation (8) and H is the ADM energy of the space-time [3]. In (11), $r = r'_+$ is the first order pole. Now the integral can be done by deforming the contour, so as to ensure that positive energy solutions decay in time (that is, into the lower half ω' plane) [3]. In this way we obtain

$$\mathrm{Im}\,S = 2\pi\int_0^{\omega}\frac{r'^2_+ - D'^2}{r'_+ - r'_-}d\omega'. \tag{12}$$

It is rather difficult to work out the integral above with respect to ω' directly. However, we can make the physical meaning clear as follows. The Bekenstein–Hawking entropy of GM black hole is

$$S_{\mathrm{BH}} = \pi(r_+^2 - D^2). \tag{13}$$

The entropy derivative of m is

$$\frac{\partial S_{\mathrm{BH}}}{\partial m} = \pi\left(2r_+\frac{\partial r_+}{\partial m} - 2D\frac{\partial D}{\partial m}\right) = 4\pi\left(\frac{r_+^2 - D^2}{r_+ - r_-}\right), \tag{14}$$

where

$$\frac{\partial r_+}{\partial m} = 1 + \frac{m - \frac{D^2}{m}}{\sqrt{m^2 + D^2 - P^2 - Q^2}}, \tag{15}$$

$$\frac{\partial D}{\partial m} = -\frac{D}{m}. \tag{16}$$

Doing the integral of m, (14) becomes to

$$\Delta S_{\mathrm{BH}} = \int_m^{m-\omega} \frac{\partial S_{\mathrm{BH}}}{\partial m'} dm' = \int_m^{m-\omega} 4\pi\left(\frac{r_+'^2 - D'^2}{r_+' - r_-'}\right) dm', \tag{17}$$

$$\Delta S_{\mathrm{BH}} = \int_0^{\omega} \frac{\partial S_{\mathrm{BH}}}{\partial m'} d\omega' = -\int_0^{\omega} 4\pi\left(\frac{r_+'^2 - D'^2}{r_+' - r_-'}\right) d\omega'. \tag{18}$$

Comparing (12) with (18), we find

$$\Delta S_{\mathrm{BH}} = -2\,\mathrm{Im}\,S. \tag{19}$$

Hence, the emission rate satisfy

$$\Gamma \to \exp(-2\,\mathrm{Im}\,S) = \exp(\Delta S_{\mathrm{BH}}). \tag{20}$$

This result is in agreement with Parikh' work. Although the constant-time slices are not flat Euclidean space, using the WKB approximation, we obtain the correct result. This shows that WKB approximation could be extended to the space which is not flat Euclidean.

3 Conclusion and Discussion

In this section, we will sum up the main points of the coordinates used to study the tunneling effect.

First, the components of the metric in the coordinates should be regular at the event horizon and there should be a time-like killing vector field, such as the Painlevè and Eddington–Finkelstein coordinates. If the time-like killing vector field does not exist, the energy conservation would not be tenable. The condition that the constant-time slices should be the flat Euclidean space is not very important. The constant-time slices in line elements (6) are not the flat Euclidean space. The calculation in Sect. 2, however, is shown that the WKB approximation is also tenable in the space which is not flat Euclidean.

Second, the event horizon and the time-like limit surface should coincide in the coordinate. We use the semi-classical WKB formula which is only tenable in the case of the infinite blue-shift near the horizon when we study the tunneling process. In addition, if only from $g_{00} = 0$ we could not obtain the event horizon, the first order pole in (11) would be at the event horizon. not at the event horizon.

Finally, in quantum mechanics, particle tunneling a barrier is a instantaneous process. So, Zhang [11] suggests that the metric in the coordinates should satisfy Landau' coordinate clock synchronization condition [14]. In fact, the line elements of the GM black hole in Eddington–Finkelstein coordinates satisfy this condition.

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